

Unifying inflation with dark energy in modified $F(R)$ Hořava-Lifshitz gravity

E. Elizalde^(a), S. Nojiri^(b,c), S. D. Odintsov^{(a,d)*} and D. Sáez-Gómez^(a)

^(a)*Consejo Superior de Investigaciones Científicas, ICE(CSIC-IEEC), Campus UAB
Facultat de Ciències, Torre C5-Parell-2a pl, E-08193 Bellaterra (Barcelona) Spain*

^(b)*Department of Physics, Nagoya University, Nagoya 464-8602, Japan*

^(c)*Kobayashi-Maskawa Institute for the Origin of Particles and
the Universe,[†] Nagoya University, Nagoya 464-8602, Japan*

^(d)*ICREA, Barcelona, Spain*

We study FRW cosmology for a non-linear modified $F(R)$ Hořava-Lifshitz gravity which has a viable convenient counterpart. A unified description of early-time inflation and late-time acceleration is possible in this theory, but the cosmological dynamic details are generically different from the ones of the convenient viable $F(R)$ model. Remarkably, for some specific choice of parameters they do coincide. The emergence of finite-time future singularities is investigated in detail. It is shown that these singularities can be cured by adding an extra, higher-derivative term, which turns out to be qualitatively different when compared with the corresponding one of the convenient $F(R)$ theory.

PACS numbers: 98.80.-k, 04.50.+h, 11.10.Wx

I. INTRODUCTION

Current observational data clearly indicates that our universe has undergone at least two periods of accelerated expansion: the early-time inflation and the present late-time cosmic acceleration. In spite of the existence of a number of (partially) successful scenarios for the inflationary and dark energy epochs, the fundamental issue of a unified description of the whole cosmic history scenario remains open. One possibility to solve this problem, relying only on the presence of gravity (and not on the introduction of additional cosmological fields), is modified gravity (for a quick presentation, see [1]). Indeed, this approach suggests a very natural unification of early-time inflation and late-time cosmic acceleration, as a purely gravitational alternative (see [2] for a review of such unified models of modified gravity). But, how general is this scenario?

The Hořava-Lifshitz quantum gravity [3] has been conjectured to be renormalizable in four dimensions, at the price of explicitly breaking Lorentz invariance. Its generalization to an $F(R)$ -formulation, which seems to be also renormalizable in $3+1$ dimensions, has been considered in Refs. [4, 5], where the Hamiltonian structure and FRW cosmology, in a power-law theory, have been investigated for such modified $F(R)$ Hořava-Lifshitz gravity. It was also conjectured there that it sustains, in principle, the possibility of a unified description of early-time inflation and the dark energy epochs.

The purpose of the present work is to study a realistic non-linear $F(R)$ gravity in the Hořava-Lifshitz formulation, with the aim to understand if such theory is in fact directly able to predict in a natural way the unification of the two acceleration eras, similarly as its done in the convenient version. It has been shown in [4] that, for a special choice of parameters, the FRW equations do coincide with the ones for the related, convenient $F(R)$ gravity. This means, in particular, that the cosmological history of such Hořava-Lifshitz $F(R)$ gravity will be just the same as for its convenient version (whereas black hole solutions are generically speaking different). For the general version of the theory the situation turns out to be more complicated. Nevertheless, the unification of inflation with dark energy is still possible and all local tests can also be passed, as we will prove.

Note that in the Hořava-Lifshitz formulation of $F(R)$ gravity, the Lorentz symmetry is explicitly broken and the restoration of the Lorentz symmetry at the observed energy scale is the main problem in this formulation. The obtained metrics in this paper are, however, FRW metrics which are almost Lorentzian at the scale of galaxy or solar system. Then when the matter sector has a Lorentz symmetry in the flat background, if the Newton law is reproduced, the violation of the Lorentz symmetry could be difficult to be observed. The restoration of the Newton law is actively investigated but in this paper, we just show a mechanism that the scalar field corresponding to the so-called scalaron

* Also at TSPU, Tomsk, Russia.

[†] See, <http://www.kmi.nagoya-u.ac.jp/index-e.html>.

in the usual $F(R)$ -gravity does not give an observable correction to the Newton law. Although there is a problem with the Lorentz symmetry breaking, the Hořava-Lifshitz $F(R)$ gravity has a much richer structure than standard $F(R)$ gravity.

The paper is organized as follows. Next section briefly reviews the formulation of modified $F(R)$ Hořava-Lifshitz gravity and its corresponding FRW cosmology. The reconstruction of the theory is presented in Sect. III. It is demonstrated that different functional forms of the theory may reproduce the same Λ CDM era, phantom (super)acceleration, or any other given cosmology. Sect. IV is devoted to the analysis of $F(R)$ Hořava-Lifshitz theories whose convenient counterparts have been proposed as viable candidates for inflation-dark energy unification. The unification is again possible but with rather different physical properties. Moreover, corrections to Newton's law are negligible in the models under consideration (Sect. V). The emergence of finite-time future singularities and their avoidance is discussed in Sect. VI. In particular, it turns out that even when the viable modified gravity is non-singular, its Hořava-Lifshitz counterpart may still remain a singular theory. Some discussions and an outlook are provided in the last section.

II. MODIFIED $F(R)$ HOŘAVA-LIFSHITZ GRAVITY

In this section, modified Hořava-Lifshitz $F(R)$ gravity is briefly reviewed [4, 5]. We start by writing a general metric in the so-called ADM decomposition in a $3+1$ spacetime (for more details see [6], [7] and references therein),

$$ds^2 = -N^2 dt^2 + g_{ij}^{(3)}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1)$$

where $i, j = 1, 2, 3$, N is the so-called lapse variable, and N^i is the shift 3-vector. In standard general relativity (GR), the Ricci scalar can be written in terms of this metric, and yields

$$R = K_{ij}K^{ij} - K^2 + R^{(3)} + 2\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu), \quad (2)$$

here $K = g^{ij}K_{ij}$, K_{ij} is the extrinsic curvature, $R^{(3)}$ is the spatial scalar curvature, and n^μ a unit vector perpendicular to a hypersurface of constant time. The extrinsic curvature K_{ij} is defined as

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij}^{(3)} - \nabla_i^{(3)} N_j - \nabla_j^{(3)} N_i \right). \quad (3)$$

In the original model [3], the lapse variable N is taken to be just time-dependent, so that the projectability condition holds and by using the foliation-preserving diffeomorphisms (6), it can be fixed to be $N = 1$. As pointed out in [31], imposing the projectability condition may cause problems with Newton's law in the Hořava gravity. On the other hand, Hamiltonian analysis shows that the non-projectable $F(R)$ -model is inconsistent [11]. For the non-projectable case, the Newton law could be restored (while keeping stability) by the “healthy” extension of the original Hořava gravity of Ref. [31].

The action for standard $F(R)$ gravity can be written as

$$S = \int d^4x \sqrt{g^{(3)}} N F(R). \quad (4)$$

Gravity of Ref. [3] is assumed to have different scaling properties of the space and time coordinates

$$x^i \rightarrow bx^i, \quad t \rightarrow b^z t, \quad (5)$$

where z is a dynamical critical exponent that renders the theory renormalizable for $z = 3$ in $3+1$ spacetime dimensions [3] (For a proposal of covariant renormalizable gravity with dynamical Lorentz symmetry breaking, see [8]). GR is recovered when $z = 1$. The scaling properties (5) render the theory invariant only under the so-called foliation-preserving diffeomorphisms:

$$\delta x^i = \zeta(x^i, t), \quad \delta t = f(t). \quad (6)$$

It has been pointed that, in the IR limit, the full diffeomorphisms are recovered, although the mechanism for this transition is not physically clear. The action considered here was introduced in Ref. [4],

$$S = \frac{1}{2\kappa^2} \int dt d^3x \sqrt{g^{(3)}} N F(\tilde{R}), \quad \tilde{R} = K_{ij}K^{ij} - \lambda K^2 + R^{(3)} + 2\mu \nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu) - L^{(3)}(g_{ij}^{(3)}), \quad (7)$$

where κ is the dimensionless gravitational coupling, and where, two new constants λ and μ appear, which account for the violation of the full diffeomorphism transformations. A degenerate version of the above $F(R)$ -theory with $\mu = 0$

has been proposed and studied in Ref. [9]. Note that in the original Hořava gravity theory [3], the third term in the expression for \tilde{R} can be omitted, as it becomes a total derivative. The term $L^{(3)}(g_{ij}^{(3)})$ is chosen to be [3]

$$L^{(3)}(g_{ij}^{(3)}) = E^{ij} G_{ijkl} E^{kl}, \quad (8)$$

where G_{ijkl} is the inverse of the generalized De Witt metric, namely

$$G^{ijkl} = \frac{1}{2} \left(g^{(3)ik} g^{(3)jl} + g^{(3)il} g^{(3)jk} \right) - \lambda g^{(3)ij} g^{(3)kl}. \quad (9)$$

Therefore we have

$$G_{ijkl} = \frac{1}{2} \left(g_{ik}^{(3)} g_{jl}^{(3)} + g_{il}^{(3)} g_{jk}^{(3)} \right) - \tilde{\lambda} g_{ij}^{(3)} g_{kl}^{(3)}, \quad \tilde{\lambda} = \frac{\lambda}{3\lambda - 1}. \quad (10)$$

Note that G^{ijkl} is singular for $\lambda = 1/3$ and therefore G_{ijkl} exist if $\lambda \neq 1/3$.

In Ref. [3], the expression for E_{ij} is constructed to satisfy the “detailed balance principle” in order to restrict the number of free parameters of the theory. This is defined through variation of an action

$$\sqrt{g^{(3)}} E^{ij} = \frac{\delta W[g_{kl}^{(3)}]}{\delta g_{ij}^{(3)}}, \quad (11)$$

where the form of $W[g_{kl}^{(3)}]$ is given in Ref. [10] for $z = 2$ and $z = 3$. Other forms for $L^{(3)}(g_{ij}^{(3)})$ have been suggested that abandons the detailed balance condition but still render the theory power-counting renormalizable (see Ref. [5]).

We are interested in the study of (accelerating) cosmological solutions for the theory described by action (7). Spatially-flat FRW metric is assumed

$$ds^2 = -N^2 dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2. \quad (12)$$

If we also assume the projectability condition, N can be taken to be just time-dependent and, by using the foliation-preserving diffeomorphisms (6), it can be fixed to be unity, $N = 1$. When we do not assume the projectability condition, N depends on both the time and spatial coordinates, first. Then, just as an assumption of the solution, N is taken to be unity.

For the metric (12), the scalar \tilde{R} is given by

$$\tilde{R} = \frac{3(1 - 3\lambda + 6\mu)H^2}{N^2} + \frac{6\mu}{N} \frac{d}{dt} \left(\frac{H}{N} \right). \quad (13)$$

For the action (7), and assuming the FRW metric (13), the second FRW equation can be obtained by varying the action with respect to the spatial metric $g_{ij}^{(3)}$, which yields

$$0 = F(\tilde{R}) - 2(1 - 3\lambda + 3\mu) \left(\dot{H} + 3H^2 \right) F'(\tilde{R}) - 2(1 - 3\lambda) \dot{\tilde{R}} F''(\tilde{R}) + 2\mu \left(\dot{\tilde{R}}^2 F^{(3)}(\tilde{R}) + \ddot{\tilde{R}} F''(\tilde{R}) \right) + \kappa^2 p_m, \quad (14)$$

here $\kappa^2 = 16\pi G$, p_m is the pressure of a perfect fluid that fills the Universe, and $N = 1$. Note that this equation becomes the usual second FRW equation for convenient $F(R)$ gravity (4), by setting the constants $\lambda = \mu = 1$. When we assume the projectability condition, variation over N of the action (7) yields the following global constraint

$$0 = \int d^3x \left[F(\tilde{R}) - 6(1 - 3\lambda + 3\mu)H^2 - 6\mu\dot{H} + 6\mu H \dot{\tilde{R}} F''(\tilde{R}) - \kappa^2 \rho_m \right]. \quad (15)$$

Now, using the ordinary conservation equation for the matter fluid $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, and integrating Eq. (14),

$$0 = F(\tilde{R}) - 6 \left[(1 - 3\lambda + 3\mu)H^2 + \mu\dot{H} \right] F'(\tilde{R}) + 6\mu H \dot{\tilde{R}} F''(\tilde{R}) - \kappa^2 \rho_m - \frac{C}{a^3}, \quad (16)$$

where C is an integration constant, taken to be zero, according to the constraint equation (15). If we do not assume the projectability condition, we can directly obtain (16), which corresponds to the first FRW equation, by variation over N . Hence, starting from a given $F(\tilde{R})$ function, and solving Eqs. (14) and (15), a cosmological solution can be obtained.

III. RECONSTRUCTING FRW COSMOLOGY IN $F(R)$ HOŘAVA-LIFSHITZ GRAVITY

To start, let us analyze the simple model $F(\tilde{R}) = \tilde{R}$, which cosmology was studied in [13] (for a complete analysis of cosmological perturbations, see [14]). In such a case, the FRW equations look similar to GR,

$$H^2 = \frac{\kappa^2}{3(3\lambda - 1)}\rho_m, \quad \dot{H} = -\frac{\kappa^2}{2(3\lambda - 1)}(\rho_m + p_m), \quad (17)$$

where, for $\lambda \rightarrow 1$, the standard FRW equations are recovered. Note that the constant μ is now irrelevant because, as pointed out above, the term in front of μ in (7) becomes a total derivative. For such theory, one has to introduce a dark energy source as well as an inflaton field, in order to reproduce the cosmic and inflationary accelerated epochs, respectively. It is also important to note that, for this case, the coupling constant is restricted to be $\lambda > 1/3$, otherwise Eqs. (17) become inconsistent. It seems reasonable to think that, for the current epoch, where \tilde{R} has a small value, the IR limit of the theory is satisfied $\lambda \sim 1$, but for the inflationary epoch, when the scalar curvature \tilde{R} goes to infinity, λ will take a different value. It has been realized that, for $\lambda = 1/3$, the theory develops an anisotropic Weyl invariance (see [3]), and thus it takes a special role, although for the present model this value is not allowed.

We now discuss some cosmological solutions of $F(\tilde{R})$ Hořava-Lifshitz gravity. The first FRW equation, given by (16) with $C = 0$, can be rewritten as a function of the number of e-foldings $\eta = \ln \frac{a}{a_0}$, instead of the usual time t . This technique has been developed in Ref. [15] for convenient $F(R)$ gravity (for review of reconstruction method in modified gravity, see [16]), where it was shown that any $F(R)$ theory can be reconstructed for a given cosmological solution. Here, we extend such formalism to the Hořava-Lifshitz $F(R)$ gravity. Since $\frac{d}{dt} = H \frac{d}{d\eta}$ and $\frac{d^2}{dt^2} = H^2 \frac{d^2}{d\eta^2} + H \frac{dH}{d\eta} \frac{d}{d\eta}$, the first FRW equation (16) is rewritten as

$$0 = F(\tilde{R}) - 6[(1 - 3\lambda + 3\mu)H^2 + \mu HH'] \frac{dF(\tilde{R})}{d\tilde{R}} + 36\mu H^2[(1 - 3\lambda + 6\mu)HH' + \mu H'^2 + \mu H''H] \frac{d^2F(\tilde{R})}{d^2\tilde{R}} - \kappa^2\rho_m, \quad (18)$$

where the primes denote derivatives with respect to η . Thus, in this case there is no restriction on the values of λ or μ . By using the energy conservation equation, and assuming a perfect fluid with equation of state (EoS) $p_m = w_m\rho_m$, the energy density yields

$$\rho_m = \rho_0 a^{-3(1+w_m)} = \rho_0 a_0^{-3(1+w_m)} e^{-3(1+w_m)\eta}. \quad (19)$$

As the Hubble parameter can be written as a function of the number of e-foldings, $H = H(\eta)$, the scalar curvature in (13) takes the form

$$\tilde{R} = 3(1 - 3\lambda + 6\mu)H^2 + 6\mu HH', \quad (20)$$

which can be solved with respect to η as $\eta = \eta(\tilde{R})$, and one gets an expression (18) that gives an equation on $F(\tilde{R})$ with the variable \tilde{R} . This can be simplified a bit by writing $G(\eta) = H^2$ instead of the Hubble parameter. In such case, the differential equation (18) yields

$$0 = F(\tilde{R}) - 6\left[(1 - 3\lambda + 3\mu)G + \frac{\mu}{2}G'\right] \frac{dF(\tilde{R})}{d\tilde{R}} + 18\mu[(1 - 3\lambda + 6\mu)GG' + \mu GG''] \frac{d^2F(\tilde{R})}{d^2\tilde{R}} - \kappa^2\rho_0 a_0^{-3(1+w)} e^{-3(1+w)\eta}, \quad (21)$$

and the scalar curvature is now written as $\tilde{R} = 3(1 - 3\lambda + 6\mu)G + 3\mu G'$. Hence, for a given cosmological solution $H^2 = G(\eta)$, one can resolve Eq. (21), and the $F(\tilde{R})$ that reproduces such solution is obtained.

As an example, we consider the Hubble parameter that reproduces the Λ CDM epoch. It is expressed as

$$H^2 = G(\eta) = H_0^2 + \frac{\kappa^2}{3}\rho_0 a^{-3} = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3} e^{-3\eta}. \quad (22)$$

where H_0 and ρ_0 are constant. In General Relativity, the terms on the rhs of Eq. (22) correspond to an effective cosmological constant $\Lambda = 3H_0^2$ and to cold dark matter with EoS parameter $w = 0$. The corresponding $F(\tilde{R})$ can be reconstructed by following the same steps as described above. Using the expression for the scalar curvature $\tilde{R} = 3(1 - 3\lambda + 6\mu)G + 3\mu G'$, the relation between \tilde{R} and η is obtained,

$$e^{-3\eta} = \frac{\tilde{R} - 3(1 - 3\lambda + 6\mu)H_0^2}{3k(1 + 3(\mu - \lambda))}, \quad (23)$$

where $k = \frac{\kappa^2}{3}\rho_0 a_0^{-3}$. Then, substituting (22) and (23) into Eq. (21), one gets the differential expression

$$0 = (1 - 3\lambda + 3\mu)F(\tilde{R}) - 2\left(1 - 3\lambda + \frac{3}{2}\mu\right)\tilde{R} + 9\mu(1 - 3\lambda)H_0^2\frac{dF(\tilde{R})}{d\tilde{R}} - 6\mu(\tilde{R} - 9\mu H_0^2)(\tilde{R} - 3H_0^2(1 - 3\lambda + 6\mu))\frac{d^2F(\tilde{R})}{d^2\tilde{R}} - R - 3(1 - 3\lambda + 6\mu)H_0^2, \quad (24)$$

where, for simplicity, we have considered a pressureless fluid $w = 0$ in Eq. (21). Performing the change of variable $x = \frac{\tilde{R} - 9\mu H_0^2}{3H_0^2(1 + 3(\mu - \lambda))}$, the homogeneous part of Eq. (24) can be easily identified as an hypergeometric differential equation

$$0 = x(1 - x)\frac{d^2F}{dx^2} + (\gamma - (\alpha + \beta + 1)x)\frac{dF}{dx} - \alpha\beta F, \quad (25)$$

with the set of parameters (α, β, γ) being given by

$$\gamma = -\frac{1}{2}, \quad \alpha + \beta = \frac{1 - 3\lambda - \frac{3}{2}\mu}{3\mu}, \quad \alpha\beta = -\frac{1 + 3(\mu - \lambda)}{6\mu}. \quad (26)$$

The complete solution of Eq. (25) is a Gauss' hypergeometric function plus a linear term and a cosmological constant coming from the particular solution of Eq. (24), namely

$$F(\tilde{R}) = C_1 F(\alpha, \beta, \gamma; x) + C_2 x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x) + \frac{1}{\kappa_1}\tilde{R} - 2\Lambda. \quad (27)$$

where C_1 and C_2 are constants, $\kappa_1 = 3\lambda - 1$ and $\Lambda = -\frac{3H_0^2(1-3\lambda+9\mu)}{2(1-3\lambda+3\mu)}$. Note that for the exact cosmology (22), the classical $F(R)$ gravity was reconstructed and studied in Refs. [15]-[17]. In this case, the solution (27) behaves similarly to the classical $F(R)$ theory, except that now the parameters of the theory depend on (λ, μ) , which are allowed to vary as it was noted above. We can conclude that the cosmic evolution described by the Hubble parameter (22) is reproduced by this class of theories.

One can also explore the solution (22) for a particular choice on the parameters $\mu = \lambda - \frac{1}{3}$, which plays a special role as it is shown below. In this case, the scalar \tilde{R} turns out to be a constant, and Eq. (24), in the presence of a pressureless fluid, has the solution

$$F(\tilde{R}) = \frac{1}{\kappa_1}\tilde{R} - 2\Lambda, \quad \text{with} \quad \Lambda = \frac{3}{2}(3\lambda - 1)H_0^2. \quad (28)$$

Hence, for this constraint on the parameters, the only consistent solution reduces to the Hořava linear theory with a cosmological constant.

As a further example, we consider the so-called phantom accelerating expansion. Currently, observational data do not totally exclude the possibility that the Universe could have already crossed the phantom divide, which means that the effective EoS for dark energy would presently be slightly less than -1 . Such kind of system can be easily expressed in GR, where the FRW equation reads $H^2 = \frac{\kappa^2}{3}\rho_{\text{ph}}$. Here the subscript “ph” denotes the phantom nature of the fluid, which has an EoS given by $p_{\text{ph}} = w_{\text{ph}}\rho_{\text{ph}}$ with $w_{\text{ph}} < -1$. By using the energy conservation equation, the solution for the Hubble parameter turns out to be

$$H(t) = \frac{H_0}{t_s - t}, \quad (29)$$

where $H_0 = -1/3(1 + w_{\text{ph}})$, and t_s is the Rip time which represents the time still remaining up to the Big Rip singularity. As in the above example, one can rewrite the Hubble parameter as a function of the number of e-foldings; this yields

$$G(\eta) = H^2(\eta) = H_0^2 e^{2\eta/H_0}. \quad (30)$$

Then, by using the expression of the scalar curvature, the relation between \tilde{R} and η is given by

$$e^{2\eta/H_0} = \frac{R}{H_0(AH_0 + 6\mu)}. \quad (31)$$

By inserting (30) and (31) into the differential equation (21), we get

$$\tilde{R}^2 \frac{d^2 F(\tilde{R})}{d\tilde{R}^2} + k_1 \tilde{R} \frac{dF(\tilde{R})}{d\tilde{R}} + k_0 F(\tilde{R}) = 0, \quad (32)$$

where

$$k_1 = -\frac{(AH_0 + 6\mu)((AH_0 + 3\mu))}{6\mu(AH_0 + 12\mu)}, \quad k_0 = \frac{(AH_0 + 6\mu)^2}{12\mu(AH_0 + 12\mu)}, \quad (33)$$

here we have neglected any kind of matter contribution for simplicity. Eq. (32) is an Euler equation, whose solution is well known

$$F(R) = C_1 R^{m_+} + C_2 R^{m_-}, \quad \text{where} \quad m_{\pm} = \frac{1 - k_1 \pm \sqrt{(k_1 - 1)^2 - 4k_0}}{2}. \quad (34)$$

Such theory belongs to the class of models with positive and negative powers of the curvature introduced in Ref. [18]. The existence of the negative power of curvature indicates that the flat Minkowski space is not realized. If the coefficient C_2 is small enough, however, the difference from the flat Minkowski space could not be observed. Moreover, one of the reasons to consider non-linear models of next section is related with the fact that Minkowski solution may be realized there.

Hence, a $F(\tilde{R})$ Hořava-Lifshitz gravity has been reconstructed that reproduces the phantom dark epoch with no need of any exotic fluid. In the same way, any given cosmology may be reconstructed.

IV. UNIFIED INFLATION AND DARK ENERGY IN MODIFIED HOŘAVA-LIFSHITZ GRAVITY

Let us consider here some viable $F(\tilde{R})$ gravities which admit the unification [18] of inflation with late-time acceleration. In the convenient $F(R)$ theory, a number of viable models (see [19], [20]) which pass all local tests and are able to unify the inflationary and the current cosmic accelerated epochs have been proposed. Here we extend this class of models to the Hořava-Lifshitz gravity. We consider the action,

$$F(\tilde{R}) = \tilde{R} + f(\tilde{R}), \quad (35)$$

where it is assumed that the term $f(\tilde{R})$ becomes important at cosmological scales, while for scales compared with the Solar system one the theory becomes linear on \tilde{R} . As an example, we consider the following function [19]

$$f(\tilde{R}) = \frac{\tilde{R}^n(\alpha\tilde{R}^n - \beta)}{1 + \gamma\tilde{R}^n}, \quad (36)$$

where (α, β, γ) are constants and $n > 1$. This theory reproduces the inflationary and cosmic acceleration epochs in convenient $F(R)$ gravity (see Ref. [19]), which is also the case in the present theory, as will be shown. During inflation, it is assumed that the curvature scalar tends to infinity. In this case the model (35), with (36), behaves as

$$\lim_{\tilde{R} \rightarrow \infty} F(\tilde{R}) = \alpha\tilde{R}^n. \quad (37)$$

Then, by solving the FRW equation (16), this kind of function yields a power-law solution of the type

$$H(t) = \frac{h_1}{t}, \quad \text{where} \quad h_1 = \frac{2\mu(n-1)(2n-1)}{1-3\lambda+6\mu-2n(1-3\lambda+3\mu)}. \quad (38)$$

This solution produces acceleration during the inflationary epoch if the parameters of the theory are properly defined. The acceleration parameter is given by $\frac{\ddot{a}}{a} = h_1(h_1 - 1)/t^2$, thus, for $h_1 > 1$ the inflationary epoch is well reproduced by the model (36). On the other hand, the function (36) has a minimum at \tilde{R}_0 , given by

$$\tilde{R}_0 \sim \left(\frac{\beta}{\alpha\gamma}\right)^{1/4}, \quad f'(\tilde{R}) = 0, \quad f(\tilde{R}) = -2\Lambda \sim -\frac{\beta}{\gamma}, \quad (39)$$

where we have imposed the condition $\beta\gamma/\alpha \gg 1$. Then, at the current epoch the scalar curvature acquires a small value which can be fixed to coincide with the minimum (39), such that the FRW equations (14) and (16) yield

$$H^2 = \frac{\kappa^2}{3(3\lambda-1)}\rho_m + \frac{2\Lambda}{3(3\lambda-1)} \quad \dot{H} = -\kappa^2 \frac{\rho_m + p_m}{3\lambda-1}, \quad (40)$$

which look very similar to the standard FRW equations in General Relativity, except for the parameter λ . For first consideration of FRW eqs. in Hořava-Lifshitz gravity based on General Relativity see ref. [12]. As has been pointed out, at the current epoch the scalar \tilde{R} is small, so the theory is in the IR limit where the parameter $\lambda \sim 1$, and the equations approach the usual ones for $F(R)$ gravity. Hence, the FRW equations (40) reproduce the behavior of the well known Λ CDM model with no need to introduce a dark energy fluid to explain the current universe acceleration.

As another example of the models described by (35), we can considered the function [20, 21],

$$f(\tilde{R}) = -\frac{(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1}}{f_0 + f_1 [(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1}]} = -\frac{1}{f_1} + \frac{f_0/f_1}{f_0 + f_1 [(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1}]} . \quad (41)$$

This function could also serve for the unification of inflation and cosmic acceleration but, in this case, when one takes the limit $\tilde{R} \rightarrow \infty$, one gets

$$\lim_{\tilde{R} \rightarrow \infty} F(\tilde{R}) = \tilde{R} - 2\Lambda_i, \quad \text{where} \quad \Lambda_i = 1/2f_1, \quad (42)$$

where the subscript i denotes that we are in the inflationary epoch. By inserting this into Eqs. (14) and (16), the FRW equations take the same form as in (40). Then, for the function (41) the inflationary epoch is produced by an effective cosmological constant, which implies that the parameter $\lambda > 1/3$, or the equations themselves will present inconsistencies, as it was discussed in the above section. For the current epoch, it is easy to see that the function (41) exhibits a minimum for $\tilde{R} = \tilde{R}_0$, which implies, as in the model above, an effective cosmological constant for late time that can produce the cosmic acceleration. The emergence of matter dominance before the dark energy epoch can be exhibited, in analogy with the case of the convenient theory. Hence, we have shown that the model (41) also unifies the cosmic expansion history, although with different properties during the inflationary epoch as compared with the model (36). This could be very important for the precise study of the evolution of the parameters of the theory.

It is also interesting to explore the de Sitter solutions allowed by the theory (7). By taking $H(t) = H_0$, the FRW equation (16), in absence of any kind of matter and with $C = 0$, reduces to

$$0 = F(\tilde{R}_0) - 6H_0^2(1 - 3\lambda + 3\mu)F'(\tilde{R}_0), \quad (43)$$

which reduces to an algebraic equation that, for an specific model, can be solved yielding the possible de Sitter points allowed by the theory. As an example, let us consider the model (36), where Eq. (43) takes the form

$$\tilde{R}_0 + \frac{\tilde{R}_0^n(\alpha\tilde{R}_0^n - \beta)}{1 + \gamma\tilde{R}_0^n} + \frac{6H_0^2(-1 + 3\lambda - 3\mu) \left[1 + n\alpha\gamma\tilde{R}_0^{3n-1} + \tilde{R}_0^{n-1}(2\gamma\tilde{R}_0 - n\beta) + \tilde{R}_0^{2n-1}(\gamma^2\tilde{R}_0 + 2n\alpha) \right]}{(1 + \gamma\tilde{R}_0^n)^2} = 0. \quad (44)$$

Here $\tilde{R}_0 = 3(1 - 3\lambda + 6\mu)H_0^2$. By specifying the free parameters of the theory, one can solve Eq. (44), which yields several de Sitter points, as the one studied above. They can be used to explain the coincidence problem, with the argument that the present will not be the only late-time accelerated epoch experienced by our Universe. In standard $F(R)$, it was found for this same model that it contains at least two de Sitter points along the cosmic history (see [22]). In the same way, the second model studied here (41), provides several de Sitter points in the course of the cosmic history. Note that when $\mu = \lambda - \frac{1}{3}$, Eq. (43) turns out to be much more simple, it reduces to $F(\tilde{R}_0) = 0$, where the de Sitter points are the roots. For example, for (44) we have $\tilde{R}_0(1 + \gamma\tilde{R}_0^n) + \tilde{R}_0^n(\alpha\tilde{R}_0^n - \beta) = 0$, where the number of positive roots (de Sitter points) depends on the free parameters of the theory.

Summing up, it has been here shown that, also in $F(\tilde{R})$ Hořava-Lifshitz gravity, the so-called viable models, as (36) or (41), can in fact reproduce the whole cosmological history of the universe, with no need to involve any extra fields or a cosmological constant. We do not, of course, believe the complicated model (36) or (41) could be a final model. These models, at best, could be low energy effective theories coming from a more beautiful final theory. Even in the usual quantum field theories, the corresponding low energy effective theories are rather complicated due to the terms induced by the quantum effects. However, our main motivation to consider precisely them is related with the fact that they pass the known local tests as usual $F(R)$ theories.

V. NEWTON LAW CORRECTIONS IN $F(\tilde{R})$ GRAVITY

As is well-known, modified gravity may lead to violations of local tests. We explore in this section how to avoid these violations of Newton's law. It is known that $F(\tilde{R})$ theories include scalar particle. This scalar field could give

rise to a fifth force and to variations of the Newton law, which can be avoided by a kind of the so-called chameleon mechanism [23]. Note that scalars with time-dependent mass were also considered in [24].

In the original Hořava gravity, the projectability condition may cause problems with the Newton law [31] but the model without the projectability condition could be inconsistent for the $F(R)$ -model [11]. The Newton law in the Hořava gravity may be restored by the “healthy” extension [31]. In this section, we do not discuss the gravity sector corresponding to the Hořava gravity but we show that the scalar mode, which also appears in the usual $F(R)$ gravity, can decouple from gravity and matter, and then the scalar mode does not give a measurable correction to Newton’s law.

To show this, we consider a function of the type (35), and rewrite action (7) as

$$S = \int dt d^3x \sqrt{g^{(3)}} N \left[(1 + f'(A))(\tilde{R} - A) + A + f(A) \right], \quad (45)$$

where A is an auxiliary scalar field. It is easy to see that variation of action (45) over A gives $A = \tilde{R}$. Performing the conformal transformation $g_{ij}^{(3)} = e^{-\phi} \tilde{g}_{ij}^{(3)}$, with $\phi = \frac{2}{3} \ln(1 + f'(A))$, action (45) yields

$$S = \int dt d^3x \sqrt{g^{(3)}} \left[\tilde{K}_{ij} \tilde{K}^{ij} - \lambda \tilde{K}^2 + \left(-\frac{1}{2} + \frac{3}{2}\lambda - \frac{3}{2}\mu \right) \dot{\tilde{g}}^{ij(3)} \tilde{g}_{ij}^{(3)} \dot{\phi} + \left(\frac{3}{4} - \frac{9}{4}\lambda + \frac{9}{2}\mu \right) \dot{\phi}^2 - V(\phi) + \tilde{L}(\tilde{g}^{(3)}, \phi) \right], \quad (46)$$

where \tilde{K}_{ij} is the extrinsic curvature given by $\tilde{g}_{ij}^{(3)}$ as in (3), $\tilde{K} = \tilde{g}^{(3)ij} \tilde{K}_{ij}$, $\tilde{L}(\tilde{g}^{(3)}, \phi)$ is the conformally transformed term in (8), and the scalar potential is

$$V(\phi) = \frac{A(\phi)f'(A(\phi)) - f(A(\phi))}{1 + f'(A(\phi))}. \quad (47)$$

Note that, differently from the convenient $F(R)$, in action (46) there is a coupling term between the scalar field ϕ and the spatial metric $\tilde{g}_{ij}^{(3)}$, which can thus be dropped, by imposing the following condition on the parameters [5]:

$$\mu = \lambda - \frac{1}{3}. \quad (48)$$

This condition also renders the theory power-counting renormalizable, for the same z as in the original Hořava model.

Let us now investigate the term (8). As already pointed out, for local scales, where the scalar curvature is assumed to be very small, the theory enters the IR limit, where such term could be written as a spatial curvature,

$$L^{(3)}(g^{(3)}, \phi) \sim R^{(3)}. \quad (49)$$

Then, corrections to the Newton law will come from the coupling that now appears between the scalar field and matter, which makes a test particle to deviate from its geodesic path, unless the mass of the scalar field is large enough (since then the effect could be very small). The precise value can be calculated from

$$m_\phi^2 = \frac{1}{2} \frac{d^2 V(\phi)}{d\phi^2} = \frac{1 + f'(A)}{f''(A)} - \frac{A + f(A)}{1 + f'(A)}. \quad (50)$$

In view of that, we can now analyze the models studied in the last section. We are interested to see the behavior at local scales, as on Earth, where the scalar curvature is around $A = \tilde{R} \sim 10^{-50} \text{eV}^2$, or in the solar system, where $A = \tilde{R} \sim 10^{-61} \text{eV}^2$. The function (36) and its derivatives can be approximated around these points as

$$f(\tilde{R}) \sim -\frac{\beta}{\gamma}, \quad f'(\tilde{R}) \sim \frac{n\alpha}{\gamma} R^{n-1}, \quad f''(\tilde{R}) \sim \frac{n(n-1)\alpha}{\gamma} R^{n-2}. \quad (51)$$

Then, the scalar mass for the model (36) is given approximately by the expression

$$m_\phi^2 \sim \frac{\gamma \tilde{R}^{2-n}}{n(n-1)\alpha}, \quad (52)$$

which becomes $m_\phi^2 \sim 10^{50n-100} \text{eV}^2$ on Earth and $m_\phi^2 \sim 10^{61n-122}$ in the Solar System. We thus see that, for $n > 2$, the scalar mass would be sufficiently large in order to avoid corrections to the Newton law. Even for the limiting case $n = 2$, the parameters γ/α can be chosen to be large enough so that any violation of the local tests is avoided.

For the model (41), the situation is quite similar. For simplicity, we impose the following condition

$$f_0 \ll f_1 \left[(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1} \right] \sim f_1 \tilde{R}^{2n+1}. \quad (53)$$

Then, function (41) and its derivatives can be written, for small values of the curvature, as

$$f(\tilde{R}) \sim -\frac{1}{f_1} + \frac{f_0}{f_1^2 \tilde{R}^{2n+1}}, \quad f'(\tilde{R}) \sim -\frac{(2n+1)f_0}{f_1^2 \tilde{R}^{2(n+1)}}, \quad f''(\tilde{R}) \sim \frac{2(2n+1)(2n+2)f_0}{f_1^2 \tilde{R}^{2n+3}}. \quad (54)$$

Using now the expression for the scalar mass (50), this yields

$$m_\phi^2 \sim \frac{f_1^2 \tilde{R}^{2n+3}}{2(2n+1)(n+1)f_0} + \frac{1}{f_1}. \quad (55)$$

Hence, as \tilde{R} is very small at solar or Earth scales ($\sim 10^{-50 \sim -61} \text{eV}^2$), and $\Lambda_i = \frac{1}{2f_1} \sim 10^{20 \sim 38} \text{eV}^2$ is the effective cosmological constant at inflation, which is much larger than the other term, it turns out that the second term on the rhs of (55) will dominate, and the scalar mass will take a value such as $m_\phi^2 \sim \frac{1}{f_1} \sim 10^{20 \sim 38} \text{eV}^2$, which is large enough as compared with the scalar curvature. As a consequence there is no observable correction to Newton's law.

We have thus shown, in all detail, that the viable models (36) and (41) do not introduce any observable correction to the Newton law at small scales. This strongly supports the choice of $F(\tilde{R})$ gravity as a realistic candidate for the unified description of the cosmological history.

VI. FINITE-TIME FUTURE SINGULARITIES IN $F(\tilde{R})$ GRAVITY

It is a well-known that a good number of effective phantom/quintessence-like dark energy models end their evolution at a finite-time future singularity. In the current section, we study the possible future evolution of the viable $F(R)$ Hořava-Lifshitz gravity considered above. It has been already proven [5] that power-law $F(R)$ Hořava-Lifshitz gravities may lead, in its evolution, to a finite-time singularity. In order to properly define the type of future singularities, let us rewrite the FRW Eqs. (14) and (16) in the following way

$$3H^2 = \frac{\kappa^2}{3\lambda - 1} \rho_{\text{eff}}, \quad -3H^2 - 2\dot{H} = \frac{\kappa^2}{3\lambda - 1} p_{\text{eff}}, \quad (56)$$

where

$$\begin{aligned} \rho_{\text{eff}} &= \frac{1}{\kappa^2 F'(\tilde{R})} \left[-F(\tilde{R}) + 3(1 - 3\lambda + 9\mu)H^2 F'(\tilde{R}) - 6\mu \dot{H} F'(\tilde{R}) - 6\mu H \dot{\tilde{R}} F''(\tilde{R}) + \kappa^2 \rho_m \right], \\ p_{\text{eff}} &= \frac{1}{\kappa^2 F'(\tilde{R})} \left[F(\tilde{R}) - (6\mu \dot{H} + 3(1 - 3\lambda + 9\mu)H^2) F'(\tilde{R}) - 2(1 - 3\lambda) \dot{\tilde{R}} F''(\tilde{R}) \right. \\ &\quad \left. + 2\mu \left[\dot{\tilde{R}}^2 F^{(3)}(\tilde{R}) + \ddot{\tilde{R}} F''(\tilde{R}) \right] + \kappa^2 p_m \right]. \end{aligned} \quad (57)$$

Then, using the expressions for the effective energy and pressure densities just defined, the list of future singularities, as classified in Ref. [25], can be extended to $F(\tilde{R})$ gravity as follows:

- Type I (“Big Rip”): For $t \rightarrow t_s$, $a \rightarrow \infty$ and $\rho_{\text{eff}} \rightarrow \infty$, $|p| \rightarrow \infty$.
- Type II (“Sudden”): For $t \rightarrow t_s$, $a \rightarrow a_s$ and $\rho_{\text{eff}} \rightarrow \rho_s$, $|p_{\text{eff}}| \rightarrow \infty$.
- Type III: For $t \rightarrow t_s$, $a \rightarrow a_s$ and $\rho_{\text{eff}} \rightarrow \infty$, $|p_{\text{eff}}| \rightarrow \infty$.
- Type IV: For $t \rightarrow t_s$, $a \rightarrow a_s$ and $\rho_{\text{eff}} \rightarrow \rho_s$, $p_{\text{eff}} \rightarrow p_s$ but higher derivatives of Hubble parameter diverge.

To illustrate the possibility of future singularities in viable $F(\tilde{R})$ gravity, we explore the model (36), which has been studied in Ref. [26] for the convenient $F(R)$ case, where it was shown that this model is non-singular, in the particular case $n = 2$. Also, it is known that for most models of $F(R)$ gravity, the future singularity can be cured by adding a term proportional to R^2 (see Ref. [27, 28]) or a non-singular modified gravity action [26]. However, this is not possible to do in the Hořava-Lifshitz gravity where, even for the simple model (36) with $n = 2$, a singularity can occur unless some restrictive conditions on the parameters are imposed.

In order to study the possible singularities that may occur from the above list, we consider a Hubble parameter close to one of such singularities, given by the following expression

$$H(t) = \frac{h_0}{(t_s - t)^q}, \quad (58)$$

where h_0 and q are constant. Depending on the value of q , the Hubble parameter (58) gives rise to a particular type of singularity. Thus, for $q \geq 1$, it gives a Big Rip singularity, for $-1 < q < 0$ a Sudden Singularity, for $0 < q < 1$ a Type III singularity and for $q < -1$, it will produce a Type IV singularity. Using the expression (13) with (58), the scalar curvature can be approximated, depending on the value of q , as

$$R \sim \begin{cases} \frac{3(1-3\lambda+6\mu)h_0^2}{(t_s-t)^{2q}} & \text{for } q > 1 \\ \frac{3(1-3\lambda+6\mu)h_0^2+6\mu q h_0}{(t_s-t)^2} & \text{for } q = 1 \\ \frac{6\mu q h_0}{(t_s-t)^{q+1}} & \text{for } q < 1 \end{cases}. \quad (59)$$

The model (36) which, as has been shown, unifies the inflationary and the dark energy epochs, makes the scalar curvature grow with time so that, close to a possible future singularity, the model can be approximated as $F(\tilde{R}) \sim \tilde{R}^n$, where $n > 1$. For the case $q > 1$, it is possible to show, from the FRW Eqs. (14) and (16), that the solution (58) is allowed for this model just in some special cases: (i) For $n = 3/2$ and $\mu = 2\lambda - \frac{2}{3}$, which contradicts the decoupling condition (48) and the Newtonian corrections (where it was found that $n > 2$). (ii) When $\mu = 0$ and $\lambda = 1/3$, which holds (48) and fixes the values of the parameters, in contradiction with the fact that fluctuations are allowed for them.

For $q = 1$, the Hubble parameter (58) is a natural solution for this model, yielding

$$H(t) = \frac{h_0}{t_s - t} \quad \text{with} \quad h_0 = \frac{2\mu(n-1)(2n-1)}{-1+3\lambda-6\mu+2n(1-3\lambda+3\mu)}. \quad (60)$$

This model allows for the possibility of occurrence of a Big Rip singularity [29], unless the parameters of the theory are fixed. As pointed out in Ref. [26], for the case $n = 2$ the Big Rip singularity can be avoided in standard $F(R)$ gravity, which can be easily seen by choosing $\lambda = \mu = 1$ in the solution (60). Nevertheless, in $F(\tilde{R})$ gravity, in order to avoid such singularity, the power n of the model has to be fixed to the value

$$n = \frac{1}{2} + \frac{3\mu}{2-6\lambda+6\mu}, \quad (61)$$

which can be interpreted as another constraint on the parameters of the theory (although, when the decoupling condition (48) is satisfied, no constraints can be imposed on n). In addition, note that $h_0 = -h_1$ in (38), where we imposed $h_1 > 1$ with the aim to reproduce the inflationary epoch, so that h_0 would be negative and the solution (60) would correspond to the Big Bang singularity. Thus, no future doomsday will take place.

For the last case, when $q < 1$, we find that the Hubble parameter (58) can be a consistent solution when Eq. (48) is satisfied and $q = (n(1-n)-1)/n(n-2)$, although if we impose $n \geq 2$, as it was found above, $q < -1$, which implies a future singularity of the type IV. The exception here is when $n = 2$, that avoids the occurrence of any type of singularity when $q < 1$. Nevertheless, even in the case that, for any reason, some kind of future singularity would be allowed in the model (36), one must also take into account possible quantum gravity effects, which may probably become important when one is close to the singularity. They have been shown, in phantom models, to prevent quite naturally the occurrence of the future singularity [30].

In summary, we have here proven that the theory (36) can actually be free of future singularities, and thus that it can make a good candidate for the unification of the cosmic history. Note that, from the corrections to the Newton law studied in the previous section, as well as from imposing avoidance of future singularities, we can fix some of the parameters of the theory. The other constant parameters that appear in (36)—which expresses the algebraic relation between the powers of the scalar curvature—could be fixed by comparing the cosmic evolution with the observed data, quite in the same way as has been successfully done for standard $F(R)$ gravity in Ref. [22].

VII. DISCUSSION

In summary, we have here investigated the FRW cosmology of a non-linear modified Hořava-Lifshitz $F(R)$ gravity theory which has a viable convenient counterpart. As shown in [4], for a special choice of the parameters, the FRW equations are just the same in both theories, and that the cosmic history of the first literally coincides with the one

for the viable $F(R)$ gravity. For a more general version of the theory, the unified description of the early-time inflation and late-time acceleration is proven to be possible too; however, the details of the cosmological dynamics are here different. Moreover, corrections to Newton's law are negligible for an extensive region of the parameter space. We have demonstrated the emergence of possible finite-time future singularities, and their avoidance, by adding extra higher-derivative terms which turn out to be qualitatively different, as compare with conventional $F(R)$ cosmology.

Using the approach of Refs. [4, 5], one can construct more complicated generalizations of modified gravity with anisotropic scaling properties. For instance, one can obtain non-local or modified Gauss-Bonnet Hořava-Lifshitz gravities. Technically, this is of course a more involved task, as compared with the $F(R)$ case. The corresponding cosmology can in principle be studied in the same way as in the present work, with expected qualitatively similar results, owing to the fact that the convenient cosmological model has been already investigated, in those cases. It is foreseen that the study of such theories will help us also in the resolution of the some problems for the theories with broken Lorentz symmetry. For instance, it is quite possible that a natural scenario of dynamical Lorentz symmetry breaking, which would reduce gravity to the Hořava limit, may be found in this way, which would be certainly interesting.

Acknowledgements

This work has been supported in part by MICINN (Spain), project FIS2006-02842, by AGAUR (Generalitat de Catalunya), contract 2009SGR-994, by Global COE Program of Nagoya University (G07) provided by the Ministry of Education, Culture, Sports, Science & Technology of Japan, and by the JSPS Grant-in-Aid for Scientific Research (S) # 22224003. DSG acknowledges an FPI grant from MICINN.

-
- [1] S. Capozziello, M. De Laurentis and V. Faraoni, arXiv:0909.4672 [gr-qc].
 - [2] S. Nojiri and S. D. Odintsov, eConf **C0602061**, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)] [arXiv:hep-th/0601213];
S. Nojiri and S. D. Odintsov, *Problems of Modern Theoretical Physics*, p.266-285, TSPU Publishing, Tomsk, arXiv:0807.0685 [hep-th].
 - [3] P. Horava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].
 - [4] M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu, arXiv:1001.4102 [hep-th].
 - [5] S. Carloni, M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu, arXiv:1003.3925 [hep-th].
 - [6] R. L. Arnowitt, S. Deser and C. W. Misner, arXiv:gr-qc/0405109;
C. Gao, Phys. Lett. B **684**, 85 (2010) [arXiv:0905.0310 [astro-ph.CO]].
 - [7] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, 1973, San Francisco.
 - [8] S. Nojiri and S. D. Odintsov, arXiv:1004.3613 [hep-th].
 - [9] J. Kluson, Phys. Rev. D **81**, 064028 (2010) [arXiv:0910.5852 [hep-th]];
J. Kluson, arXiv:1002.4859 [hep-th];
J. Kluson, JHEP **0911** (2009) 078 [arXiv:0907.3566 [hep-th]].
 - [10] P. Horava, JHEP **0903**, 020 (2009) [arXiv:0812.4287 [hep-th]].
 - [11] M. Chaichian, M. Oksanen, A. Tureanu, arXiv:1006.3235 [hep-th].
 - [12] E. Kiritsis and G. Kofinas, Nucl. Phys. B **821**, 467 (2009) [arXiv:0904.1334 [hep-th]].
 - [13] T. Takahashi and J. Soda, Phys. Rev. Lett. **102**, 231301 (2009) [arXiv:0904.0554 [hep-th]];
R. Brandenberger, Phys. Rev. D **80**, 043516 (2009) [arXiv:0904.2835 [hep-th]];
S. Mukohyama, K. Nakayama, F. Takahashi and S. Yokoyama, Phys. Lett. B **679**, 6 (2009) [arXiv:0905.0055 [hep-th]];
T. P. Sotiriou, M. Visser and S. Weinfurtner, JHEP **0910**, 033 (2009) [arXiv:0905.2798 [hep-th]];
E. N. Saridakis, Eur. Phys. J. C **67**, 229 (2010) [arXiv:0905.3532 [hep-th]];
M. Minamitsuji, Phys. Lett. B **684**, 194 (2010) [arXiv:0905.3892 [astro-ph.CO]];
G. Calcagni, Phys. Rev. D **81**, 044006 (2010) [arXiv:0905.3740 [hep-th]];
A. Wang and Y. Wu, JCAP **0907**, 012 (2009) [arXiv:0905.4117 [hep-th]];
M. i. Park, JHEP **0909**, 123 (2009) [arXiv:0905.4480 [hep-th]];
S. Nojiri and S. D. Odintsov, Phys. Rev. D **81**, 043001 (2010) [arXiv:0905.4213 [hep-th]];
M. Jamil, E. N. Saridakis and M. R. Setare, Phys. Lett. B **679**, 172 (2009) [arXiv:0906.2847 [hep-th]];
M. i. Park, JCAP **1001**, 001 (2010) [arXiv:0906.4275 [hep-th]];
C. Bogdanos and E. N. Saridakis, Class. Quant. Grav. **27**, 075005 (2010) [arXiv:0907.1636 [hep-th]];
C. G. Boehmer and F. S. N. Lobo, arXiv:0909.3986 [gr-qc];
I. Bakas, F. Bourliot, D. Lust and M. Petropoulos, Class. Quant. Grav. **27**, 045013 (2010) [arXiv:0911.2665 [hep-th]];
G. Calcagni, JHEP **0909**, 112 (2009) [arXiv:0904.0829 [hep-th]];
S. Carloni, E. Elizalde and P. J. Silva, Class. Quant. Grav. **27**, 045004 (2010) [arXiv:0909.2219 [hep-th]];

- X. Gao, Y. Wang, W. Xue and R. Brandenberger, JCAP **1002**, 020 (2010) [arXiv:0911.3196 [hep-th]];
Y. S. Myung, Y. W. Kim, W. S. Son and Y. J. Park, arXiv:0911.2525 [gr-qc];
E. J. Son and W. Kim, arXiv:1003.3055 [hep-th];
A. Wang, arXiv:1003.5152 [hep-th];
A. Ali, S. Dutta, E. N. Saridakis and A. A. Sen, arXiv:1004.2474 [astro-ph.CO].
- [14] J. O. Gong, S. Koh and M. Sasaki, Phys. Rev. D **81**, 084053 (2010) [arXiv:1002.1429 [hep-th]].
- [15] S. Nojiri, S. D. Odintsov and D. Saez-Gomez, Phys. Lett. B **681**, 74 (2009) [arXiv:0908.1269 [hep-th]].
- [16] S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006) [arXiv:hep-th/0608008];
S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, Phys. Lett. B **639**, 135 (2006) [arXiv:astro-ph/0604431];
S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006) [arXiv:hep-th/0608008];
S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. **66**, 012005 (2007) [arXiv:hep-th/0611071].
- [17] P. K. S. Dunsby, E. Elizalde, R. Goswami, S. Odintsov and D. S. Gomez, arXiv:1005.2205 [gr-qc].
- [18] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003) [arXiv:hep-th/0307288].
- [19] S. Nojiri and S. D. Odintsov, Phys. Rev. D **77**, 026007 (2008) [arXiv:0710.1738 [hep-th]].
- [20] S. Nojiri and S. D. Odintsov, Phys. Lett. B **657**, 238 (2007) [arXiv:0707.1941 [hep-th]];
G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **77**, 046009 (2008) [arXiv:0712.4017 [hep-th]].
- [21] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007) [arXiv:0705.1158 [astro-ph]].
- [22] E. Elizalde and D. Saez-Gomez, Phys. Rev. D **80**, 044030 (2009) [arXiv:0903.2732 [hep-th]].
- [23] J. Khoury and A. Weltman, Phys. Rev. Lett. **93**, 171104 (2004) [arXiv:astro-ph/0309300];
J. Khoury and A. Weltman, Phys. Rev. D **69**, 044026 (2004) [arXiv:astro-ph/0309411].
- [24] D. F. Mota and J. D. Barrow, Phys. Lett. B **581**, 141 (2004) [arXiv:astro-ph/0306047].
- [25] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005) [arXiv:hep-th/0501025].
- [26] S. Nojiri and S. D. Odintsov, arXiv:0910.1464 [hep-th].
- [27] M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **22**, L35 (2005) [arXiv:hep-th/0409177].
- [28] S. Nojiri and S. D. Odintsov, Phys. Rev. D **78**, 046006 (2008) [arXiv:0804.3519 [hep-th]];
K. Bamba, S. Nojiri and S. D. Odintsov, JCAP **0810**, 045 (2008) [arXiv:0807.2575 [hep-th]];
S. Capozziello, M. De Laurentis, S. Nojiri and S. D. Odintsov, Phys. Rev. D **79**, 124007 (2009) [arXiv:0903.2753 [hep-th]].
- [29] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002) [arXiv:astro-ph/9908168];
R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003) [arXiv:astro-ph/0302506];
B. McInnes, JHEP **0208** (2002) 029 [arXiv:hep-th/0112066];
S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003) [arXiv:hep-th/0303117];
S. Nojiri and S. D. Odintsov, Phys. Lett. B **571**, 1 (2003) [arXiv:hep-th/0306212];
S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005) [arXiv:hep-th/0505215];
V. Faraoni, Int. J. Mod. Phys. D **11**, 471 (2002) [arXiv:astro-ph/0110067];
P. F. Gonzalez-Diaz, Phys. Lett. B **586**, 1 (2004) [arXiv:astro-ph/0312579];
M. Sami and A. Toporensky, Mod. Phys. Lett. A **19**, 1509 (2004) [arXiv:gr-qc/0312009];
H. Stefancic, Phys. Lett. B **586**, 5 (2004) [arXiv:astro-ph/0310904];
L. P. Chimento and R. Lazkoz, Phys. Rev. Lett. **91**, 211301 (2003) [arXiv:gr-qc/0307111];
Mod. Phys. Lett. A **19**, 2479 (2004) [arXiv:gr-qc/0405020];
J. G. Hao and X. Z. Li, Phys. Lett. B **606**, 7 (2005) [arXiv:astro-ph/0404154];
E. Babichev, V. Dokuchaev and Yu. Eroshenko, Class. Quant. Grav. **22**, 143 (2005) [arXiv:astro-ph/0407190];
X. F. Zhang, H. Li, Y. S. Piao and X. M. Zhang, Mod. Phys. Lett. A **21**, 231 (2006) [arXiv:astro-ph/0501652];
E. Elizalde, S. Nojiri, S. D. Odintsov and P. Wang, Phys. Rev. D **71**, 103504 (2005) [arXiv:hep-th/0502082];
M. P. Dabrowski and T. Stachowiak, Annals Phys. **321**, 771 (2006) [arXiv:hep-th/0411199];
F. S. N. Lobo, Phys. Rev. D **71**, 084011 (2005) [arXiv:gr-qc/0502099];
R. G. Cai, H. S. Zhang and A. Wang, Commun. Theor. Phys. **44**, 948 (2005) [arXiv:hep-th/0505186];
I. Y. Arefeva, A. S. Koshelev and S. Y. Vernov, Phys. Rev. D **72**, 064017 (2005) [arXiv:astro-ph/0507067];
H. Q. Lu, Z. G. Huang and W. Fang, arXiv:hep-th/0504038;
W. Godlowski and M. Szydlowski, Phys. Lett. B **623**, 10 (2005) [arXiv:astro-ph/0507322];
J. Sola and H. Stefancic, Phys. Lett. B **624**, 147 (2005) [arXiv:astro-ph/0505133];
B. Guberina, R. Horvat and H. Nikolic, Phys. Rev. D **72**, 125011 (2005) [arXiv:astro-ph/0507666].
- [30] E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004) [arXiv:hep-th/0405034].
- [31] D. Blas, O. Pujolas and S. Sibiryakov, Phys. Rev. Lett. **104**, 181302 (2010) [arXiv:0909.3525 [hep-th]].